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Two Dimensional Harmonic Convex and Harmonic Quasi-convex functions and their Fundamental Properties

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Abstract - The objective of this paper is to introduce the notion of two-dimensional harmonic convex and harmonic quasi-convex functions. We have defined some fundamental properties of two dimensional harmonic convex and harmonic quasi convex functions. In this paper, we have proved that product of two dimensional harmonic convex functions are two dimensional harmonic convex functions. The intersection of two dimensional harmonic convex sets is two-dimensional harmonic convex set. It is, further proved that a two dimensional harmonically log-convex functions imply two dimensional harmonically convex functions and two dimensional harmonically convex functions implies the two dimensional harmonically quasi-convex functions. Beside this, we have also defined some fundamental properties of harmonic convex and harmonic quasi convex functions which holds in one dimension are also valid in two dimensions. The ideas and techniques used in this paper are very interesting and may be helpful for researcher to carry further research in this field.

Keywords: Two-dimensional convex set, two-dimensional Harmonic Convex set, two dimensional convex functions and two dimensional harmonic quasi convex functions.

I. Introduction

The idea of convexity is not new one even it is found in the works of Archimedes' treatment of orbit length in some other form. Hermann Minkowski and Werner Fenchel were the great mathematician, who first of all studied some geometric properties of convex sets and convex functions before 1960. At the beginning of 1960, R. Tyrrell Rockafellar and Jean-Jacques Morreau initiated a systematic study of convex analysis. Nowadays, the application of several works on convexity can be directly or indirectly seen in various subjects like real analysis, functional analysis, linear algebra, topology, non linear programming and differential geometry. Convexity theory has appeared as a powerful technique to study a wide class of unrelated problems in pure and applied sciences. Many articles have been written by a number of mathematicians on convex functions and inequalities for their different classes. In the last few decades, the subject of convex analysis got rapid development because of its geometry and its role in the optimization. There are several books devoted to different aspects of convex analysis and optimization. See [1–6]. A significant class of convex functions, called harmonic convex was introduced by Anderson et al. [1] and Iscan [4]. Noor and Noor [6, 7] have shown that the optimality conditions of the differentiable harmonic convex functions on the harmonic convex set can be expressed by a class of variational inequalities, which is called the harmonic variational inequality. For recent developments and applications, see [5-7, 8-13, 19-20]. In this paper, we show that two dimensional harmonic convex and two dimensional harmonic quasi convex functions have some nice properties. Further, we have investigated various fundamental properties of two dimensional harmonic convex functions and two dimensional harmonic quasi convex functions by taking up the same ideas and techniques used in one dimension

II. Preliminaries

In this section, we recall definitions of convex set, convex functions, harmonic convex functions, harmonic quasi convex functions and some fundamental results already defined in one dimension. By using the idea of generalizations and extensions, we have given new definitions as two-dimensional convex set, two dimensional convex functions, two dimensional harmonic convex functions and two dimensional harmonic quasi convex functions by amplifying the idea of one dimension to two dimensions.

Definition 2.1. A set $K \subseteq \mathbb{R}^2$ is said to be two dimensional convex set, if $\forall (x,y),(z,w) \in K$, we have $\left(tx+(1-t)z,\ ty+(1-t)w\right) \in K, \forall t \in [0,1]$.

Definition 2.2. A function $f: K \subset \mathbb{R}^2 \to \mathbb{R}$ is said to be two dimensional convex function, if

$$f\left(tx+(1-t)z,\ ty+(1-t)w\right) \leq tf(x,y)+(1-t)f(z,w), \forall (x,y),(z,w) \in \Delta \,\&\, t \in [0,1].$$

Definition 2.3. A set $\Delta \subseteq \mathbb{R}^2_+$ such that $\mathbb{R}^2_+ = \{(x,y): x >, y > 0\}$. Then Δ is said to be two dimensional harmonic convex set, if $\left(\frac{xz}{z+t(x-z)}, \frac{yw}{w+t(y-w)}\right) \in \Delta, \forall (x,y),(z,w) \in \Delta, t \in [0,1]$.

Definition 2.4. [8] A function $f: \Delta \subseteq \mathbb{R}^2_+ \to \mathbb{R}$ is said to be two dimensional harmonic convex function on Δ if

$$f\left(\frac{xz}{z+t(x-z)}, \frac{yw}{w+t(y-w)}\right) \leq (1-t)f(x,y)+tf(z,w)\ \forall (x,y),(z,w) \in \Delta, t \in [0,1].$$

as

convex function on Δ if

$$f\left(\frac{xz}{z+t(x-z)}, \frac{yw}{w+t(y-w)}\right) \leq \max\{f(x,y), f(z,w)\}, \forall (x,y),(z,w) \in \Delta, t \in [0,1].$$

Definition 2.7. The function f is said to be two-dimensional harmonic quasi-concave if and only if $-f$ is two dimensional harmonic quasi-convex.

(a) A function f is two dimensional harmonic quasi-convex, if whenever

$f(z,w) \geq f(x,y)$

(b) A function f is said to be strictly two dimensional harmonic quasi-convex, if

$f(z,w) > f(x,y)$

Definition2.8. [3] A function $f:\Delta \subseteq \mathbb{R}^2_+ \to \mathbb{R}$ is said to be two dimensional harmonic log-convex functions on Δ if

$$f\left(\frac{xz}{z+t(x-z)}, \frac{yw}{w+t(y-w)}\right) \leq \left((f(x,y))^{1-t}.(f(z,w))^t\right),\ \forall (x,y),(z,w) \in \Delta, t \in [0,1].$$

Definition 2.9. Let Δ be a non empty set in $\mathbb{R}^2_+$ and $f:\Delta \subseteq \mathbb{R}^2_+ \to \mathbb{R}$ be a function. Then epigraph of f denoted by $E(f)$ and is defined as

$$E(f) = \{(\xi, \lambda) : \xi = (x,y) \in \Delta, \lambda \in \mathbb{R}, f(\xi) \leq \lambda\}$$

Definition 2.10. A function $f:\Delta \subseteq \mathbb{R}^2_+ \to \mathbb{R}$ is said to be two dimensional harmonic pseudo-convex function with respect toa strictly positive function $\eta(.,.)$ such that

$f(x,y) > f(z,w)$

$$\Rightarrow f\left(\frac{xz}{z+t(x-z)}, \frac{yw}{w+t(y-w)}\right) < f(x,y) + t(1-t)\eta\left[(x,y),(z,w)\right], \forall (x,y),(z,w) \in \Delta, t \in (0,1).$$

Theorem 2.11. Let Δ be a harmonic convex set and $f:\Delta \subseteq \mathbb{R}^2_+ \to \mathbb{R}$ be a two dimensional harmonic convex function. Thenany local minimum of f is a global minimum.

Proof. Let $(x,y) \in \Delta$ be a local minimum of two-dimensional harmonic convex function *f*.

Suppose on the contrary that $f(z,w) < f(x,y) \in S$, since f is two-dimensional harmonic convex function. Then,

$$f\left(\frac{xz}{z+t(x-z)}, \frac{yw}{w+t(y-w)}\right) \leq (1-t)f(x,y) + tf(z,w), \forall (x,y),(z,w) \in \Delta, t \in [0,1].$$

$$\leq f(x,y) - tf(x,y) + tf(z,w)$$

$$= f(x,y) + t(f(z,w) - f(x,y))$$

$$\therefore f\left(\frac{xz}{z+t(x-z)}, \frac{yw}{w+t(y-w)}\right) - f(x,y) \leq t[f(z,w) - f(x,y)]$$

For some t > 0, it follows that $f\left(\frac{xz}{z+t(x-z)}, \frac{yw}{w+t(y-w)}\right) < f(x,y)$, which is a Contradiction.

Hence every local minimum of f is global minimum.

Theorem 2.12. *If* $f:\Delta \subseteq \mathbb{R}^2_+ \to \mathbb{R}$ be two dimensional harmonic log convex function on Δ, then f is two dimensional harmonic convex function implies f is two dimensional harmonic quasi-convex function.

Proof. Suppose f is two-dimensional harmonic log convex function. Then for all $(x,y),(z,w)\in\Delta$ and $t\in[0,1]$, we have

$$f\left(\frac{xz}{z+t(x-z)},\frac{yw}{w+t(y-w)}\right)\le\left(f(x,y)\right)^{1-t}.\left(f(z,w)\right)^{t}$$

$$\Rightarrow f\left(\frac{xz}{z+t(x-z)},\frac{yw}{w+t(y-w)}\right)\le\left(f(x,y)\right)^{1-t}+\left(f(z,w)\right)^{t}$$

$$\Rightarrow f\left(\frac{xz}{z+t(x-z)},\frac{yw}{w+t(y-w)}\right)\le\left(f(x,y)\right)^{1-t}+\left(f(z,w)\right)^{t}$$

$$\Rightarrow f\left(\frac{xz}{z+t(x-z)},\frac{yw}{w+t(y-w)}\right)\le(1-t)f(x,y)+tf(z,w)$$

$$\Rightarrow f\left(\frac{xz}{z+t(x-z)},\frac{yw}{w+t(y-w)}\right)\le\max\{f(x,y),f(z,w)\}$$

This proves that f is two-dimensional harmonic log convex functions

$\Rightarrow f$ is two-dimensional harmonic convex function

$\Rightarrow f$ Is two-dimensional harmonic quasi convex function

The converse of the theorem (2.12) need not be true.

III. Main Result

In this section, we discuss some properties of two dimensional harmonic convex functions and harmonic quasi convexfunctions.

Theorem3.1. If Δ_1 and Δ_2 are two two-dimensional harmonic convex sets, then $\Delta_1\cap\Delta_2$ is also a two-dimensional harmonic convex set.

Proof. Let $(x,y),(z,w)\in\Delta_1\cap\Delta_2, t\in[0,1]$. We have to prove that $\Delta_1\cap\Delta_2$ *is also* harmonic convex set. For this, $(x,y),(z,w)\in\Delta_1\cap\Delta_2$

$$\Rightarrow (x,y),(z,w)\in\Delta_1 \; and \; (x,y),(z,w)\in\Delta_2$$

$$\Rightarrow\left(\frac{xz}{z+t(x-z)},\frac{yw}{w+t(y-w)}\right)\in\Delta_1 \text{ and } \left(\frac{xz}{z+t(x-z)},\frac{yw}{w+t(y-w)}\right)\in\Delta_2$$

$$\Rightarrow\left(\frac{xz}{z+t(x-z)},\frac{yw}{w+t(y-w)}\right)\in\Delta_1\cap\Delta_2, t\in[0,1]$$

$\Rightarrow \Delta_1\cap\Delta_2$ is a two-dimensional harmonic convex set.

Theorem3.2. Let Δ be a two-dimensional harmonic convex set and $f:\Delta\subseteq \mathbb{R}^2_+\to\mathbb{R}$ be a two-dimensional harmonic convex function. Then $F=\lambda f$ is also two-dimensional harmonic convex function, where $\lambda\geq 0$.

Proof. Let Δ be a two-dimensional harmonic convex set. Then for $(x,y),(z,w)\in\Delta, t\in[0,1]$, we have

$$f\left(\frac{xz}{z+t(x-z)},\frac{yw}{w+t(y-w)}\right)=\lambda\left(\frac{xz}{z+t(x-z)},\frac{yw}{w+t(y-w)}\right)$$
$$\leq\lambda\{(1-t)f(x,y)+tf(z,w)\}$$
$$=(1-t)\lambda f(x,y)+t\lambda f(z,w)$$
$$=(1-t)f(x,y)+tf(z,w)\quad[\because\ \lambda f=f]$$

$$\left(\frac{xz}{z+t(x-z)},\frac{yw}{w+t(y-w)}\right)\leq(1-t)f(x,y)+tf(z,w),\forall(x,y),(z,w)\in\Delta,t\in[0,1].$$

$\therefore f=\lambda f$ is a two-dimensional harmonic convex function.

Theorem 3.3. Let $f:\Delta\subseteq\mathbb{R}^2_+\to\mathbb{R}$ be a two dimensional harmonic convex function on two dimensional harmonic convex set Δ . Then the level set $\Delta_\lambda=\{(x,y)\in\Delta: f(x,y)\leq\lambda,\lambda\in\mathbb{R}\}$ is a two dimensional harmonic convex set.

Proof. Let $(x,y),(z,w)\in\Delta_\lambda$. Then $f(x,y)\leq\lambda, f(z,w)\leq\lambda$.

Now $$f\left(\frac{xz}{z+t(x-z)},\frac{yw}{w+t(y-w)}\right)\leq(1-t)f(x,y)+tf(z,w)$$
$$\leq(1-t)\lambda+t\lambda$$
$$=\lambda-t\lambda+t\lambda$$
$$=\lambda$$

$$\therefore f\left(\frac{xz}{z+t(x-z)},\frac{yw}{w+t(y-w)}\right)\leq\lambda,\ \forall(x,y),(z,w)\in\Delta_\lambda$$

$\Rightarrow\Delta_\lambda$ is a two-dimensional harmonic convex set.

Theorem 3.4. The function $f:\Delta\subseteq\mathbb{R}^2_+\to\mathbb{R}$ is two dimensional harmonic convex if and only if E(f) is two dimensional harmonic convex set.

Proof. First, suppose that f is two-dimensional harmonic convex function and Let $(\xi_1,\lambda_1),(\xi_2,\lambda_2)\in E(f)$, where $\xi_1=(x,y),\xi_2=(z,w)$

Then $f(\xi_1)\leq\lambda_1, f(\xi_2)\leq\lambda_2$. For $t\in[0,1]$, we have

$$f\left(\frac{xz}{z+t(x-z)},\frac{yw}{w+t(y-w)}\right)\leq(1-t)f(x,y)+tf(z,w)$$
$$=(1-t)f(\xi_1)+tf(\xi_2)$$
$$\leq(1-t)\lambda_1+t\lambda_2$$
$$\Rightarrow\left(\left(\frac{xz}{z+t(x-z)},\frac{yw}{w+t(y-w)}\right),(1-t)\lambda_1+t\lambda_2\right)\in E(f)$$

$\Rightarrow E(f)$ is two-dimensional harmonic convex set

Conversely, suppose $E(f)$ is two-dimensional harmonic convex set and let $\xi_1, \xi_2 \in \Delta$.

Then $(\xi_1, f(\xi_1)),\ (\xi_2, f(\xi_2)) \in E(f)$ *, we have*

$$\left(\left(\frac{xz}{z+t(x-z)}, \frac{yw}{w+t(y-w)}\right), (1-t)\lambda_1 + t\lambda_2 \right) \in E(f)$$

$$\Rightarrow f\left(\frac{xz}{z+t(x-z)}, \frac{yw}{w+t(y-w)}\right) \le (1-t) f(x,y) + tf(z,w)$$

$\Rightarrow f$ is two-dimensional harmonic convex function.

Theorem 3.5. *[8] If f and ψ are two, two dimensional harmonically convex functions. If f and ψ are similarly ordered, then $f\ \psi$ is also two dimensional harmonically convex function.*

Proof. Let f and ψ are two dimensional harmonically convex functions. Then

$$f\left(\frac{xz}{z+t(x-z)}, \frac{yw}{w+t(y-w)}\right)\psi\left(\frac{xz}{z+t(x-z)}, \frac{yw}{w+t(y-w)}\right)$$

$$\le [(1-t) f(x,y) + tf(z,w)][(1-t)\psi(x,y) + t\psi(z,w)]$$

$$= (1-t)^2 f(x,y)\psi(x,y) + t(1-t) f(x,y)\psi(z,w) + t(1-t) f(z,w)\psi(x,y) + t^2 f(z,w)\psi(z,w)$$

$$= (1-t) f(x,y)\psi(x,y) + tf(z,w)\psi(z,w) + (1-t)^2 f(x,y)\psi(x,y) + t^2 f(z,w)\psi(z,w)$$

$$+t(1-t)[f(x,y)\psi(z,w) + f(z,w)\psi(x,y)] - (1-t) f(x,y)\psi(x,y) - tf(z,w)\psi(z,w)$$

$$\le (1-t) f(x,y)\psi(x,y) + tf(z,w)\psi(z,w)$$

$$f\left(\frac{xz}{z+t(x-z)}, \frac{yw}{w+t(y-w)}\right)\psi\left(\frac{xz}{z+t(x-z)}, \frac{yw}{w+t(y-w)}\right)$$

$$\le (1-t) f(x,y)\psi(x,y) + tf(z,w)\psi(z,w)$$

This proves that product of two, two dimensional harmonically convex functions is two dimensional harmonically convex functions.

Theorem 3.6. Let Δ be a two dimensional harmonic convex set. If the function $f : \Delta \subseteq \mathbb{R}^2_+ \to \mathbb{R}$ is two dimensional harmonic convex function and $\psi : \mathbb{R}^2 \to \mathbb{R}$ is linear function, then $\psi o f$ is two dimensional harmonic convex function.

Proof. Suppose that f is two-dimensional harmonic convex function and ψ is linear function. Then

$$(\psi o f)\left(\frac{xz}{z+t(x-z)}, \frac{yw}{w+t(y-w)}\right) = \psi\left[f\left(\frac{xz}{z+t(x-z)}, \frac{yw}{w+t(y-w)}\right)\right]$$

$$\le \psi\{(1-t) f(x,y) + tf(z,w)\}$$

$$= (1-t)\psi(f(x,y)) + t\psi(f(z,w))$$

$$= (1-t)(\psi o f)(x,y)) + t(\psi o f)(z,w))$$

This proves that $\psi o f$ is two-dimensional harmonic convex function

Theorem 3.7. *If $f:\Delta\subseteq\mathbb{R}^2_+\to\mathbb{R}$ be a two dimensional harmonic quasi- convex function and $\psi:\mathbb{R}^2\to\mathbb{R}$ is increasing function, then $\psi o f:\Delta\to\mathbb{R}$ is a harmonic quasi convex function.*

Proof. Suppose f is two-dimensional harmonic quasi-convex function and ψ is increasing function. Then

$$\begin{aligned}(\psi o f)\left(\frac{xz}{z+t(x-z)},\frac{yw}{w+t(y-w)}\right) &= \psi\left[f\left(\frac{xz}{z+t(x-z)},\frac{yw}{w+t(y-w)}\right)\right]\\ &\le \psi\left[\max\{f(x,y),f(z,w)\}\right]\\ &= \max\{\psi o f(x,y),\psi o f(z,w)\}\\ &= \max\{(\psi o f)(x,y),(\psi o f)(z,w)\}\end{aligned}$$

$\therefore \psi o f$ is two-dimensional harmonic quasi convex function.

Theorem 3.8. Let $f:\Delta\subseteq\mathbb{R}^2_+\to\mathbb{R}$ be a two dimensional harmonic convex function. If $\mu=\inf_{(x,y)\in\Delta} f(x,y)$, then the set $E=\{(x,y)\in\Delta: f(x,y)=\mu\}$ is a two dimensional harmonic convex set. If f is strictly two dimensional harmonic convex function, then E is a singleton.

Proof. Let $(x,y),(z,w)\in E$. Since f is harmonic convex function, therefore

$$f\left(\frac{xz}{z+t(x-z)},\frac{yw}{w+t(y-w)}\right)\le(1-t)f(x,y)+tf(z,w)=\mu,$$

$$\Rightarrow\left(\frac{xz}{z+t(x-z)},\frac{yw}{w+t(y-w)}\right)\in E$$ and hence E is a harmonic convex set.

For the second part, assume that $f(x,y)=f(z,w)=\mu$.

Since Δ is a harmonic convex set, so for $t\in[0,1]$, $\left(\frac{xz}{z+t(x-z)},\frac{yw}{w+t(y-w)}\right)\in\Delta.$

Further, since f is strictly two-dimensional harmonic convex function, so

$f\left(\frac{xz}{z+t(x-z)},\frac{yw}{w+t(y-w)}\right) <(1-t)f(x,y)+tf(z,w)=\mu$. This contradicts that $\mu=\inf_{(x,y)\in\Delta} f(x,y)$ and hence E is singleton.

Theorem 3.9. If $f:\Delta\subseteq\mathbb{R}^2_+\to\mathbb{R}$ be a two dimensional harmonic convex function such that f is two dimensional harmonic pseudo convex function with respect to strictly positive norm function $\eta(.,.)$.

Proof. Suppose $f(x,y)>f(z,w)$ and f is two-dimensional harmonic convex function. Then

$$\begin{aligned}f\left(\frac{xz}{z+t(x-z)},\frac{yw}{w+t(y-w)}\right) &\le (1-t)f(x,y)+tf(z,w)\\ &= f(x,y)-tf(x,y)+tf(z,w)\\ &= f(x,y)+t[f(z,w)-f(x,y)]\\ &< f(x,y)+t(1-t)[f(z,w)-f(x,y)]\\ &= f(x,y)+t(t-1)[f(x,y)-f(z,w)]\end{aligned}$$

$$< f(x,y) + t(t-1)\eta(u,v)$$

$$\because \eta(u,v) = f(x,y) - f(z,w) > 0$$

IV. Conclusion

In this paper, we have introduced and studied a new class of two dimensional harmonic convex and harmonic quasi convex functions. We have also derived some basic results and properties involving two dimensional harmonic convex and harmonic quasi convex functions.

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